Exceptional Confinement in $G(2)$ Gauge Theory

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Abstract

We study theories with the exceptional gauge group $G(2)$. The 14 adjoint "gluons" of a $G(2)$ gauge theory transform as $\{3\}$, $\{\overline{3}\}$ and $\{8\}$ under the subgroup $SU(3)$, and hence have the color quantum numbers of ordinary quarks, anti-quarks and gluons in QCD. Since $G(2)$ has a trivial center, a "quark" in the $\{7\}$ representation of $G(2)$ can be screened by "gluons". As a result, in $G(2)$ Yang-Mills theory the string between a pair of static "quarks" can break. In $G(2)$ QCD there is a hybrid consisting of one "quark" and three "gluons". In supersymmetric $G(2)$ Yang-Mills theory with a $\{14\}$ Majorana "gluino" the chiral symmetry is $\mathbb{Z}(4)_\chi$. Chiral symmetry breaking gives rise to distinct confined phases separated by confined-confined domain walls. A scalar Higgs field in the $\{7\}$ representation breaks $G(2)$ to $SU(3)$ and allows us to interpolate between theories with exceptional and ordinary confinement. We also present strong coupling lattice calculations that reveal basic features of $G(2)$ confinement. Just as in QCD, where dynamical quarks break the $\mathbb{Z}(3)$ symmetry explicitly, $G(2)$ gauge theories confine even without a center. However, there is not necessarily a deconfinement phase transition at finite temperature.

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1 Introduction

Understanding confinement and the dynamical mechanism behind it is a big challenge in strong interaction physics. In $SU(3)$ Yang-Mills theory confinement is associated with the $\mathbb{Z}(3)$ center of the gauge group. Since the center symmetry is unbroken at low temperatures, an unbreakable string confines static quarks in the fundamental $\{3\}$ representation to static anti-quarks in the $\{\overline{3}\}$ representation. In the high-temperature deconfined phase the Polyakov loop [1, 2] gets a non-zero expectation value and the $\mathbb{Z}(3)$ symmetry breaks spontaneously. As a result, there are three distinct deconfined phases. Potential universal behavior at the deconfinement phase transition is described by an effective 3-d scalar field theory for the Polyakov loop [3]. The center symmetry and its spontaneous breakdown were recently reviewed in [4, 5]. In full QCD the $\mathbb{Z}(3)$ symmetry is explicitly broken because quarks transform non-trivially under the center. As a result, the string connecting a quark and an anti-quark can break via the creation of dynamical quark-anti-quark pairs. Still, color remains confined and non-Abelian charged states — like single quarks or gluons — cannot exist.

In this article we ask how confinement arises in a theory whose gauge group has a trivial center. The simplest group with this property is $SO(3) = SU(2)/\mathbb{Z}(2)$. While $SO(3)$ has a trivial center, its universal covering group $SU(2)$ has the non-trivial center $\mathbb{Z}(2)$. Similarly, $SU(N_c)/\mathbb{Z}(N_c)$ has a trivial center and the corresponding universal covering group $SU(N_c)$ has the non-trivial center $\mathbb{Z}(N_c)$. When one formulates Yang-Mills theories on the lattice, one usually works with Wilson parallel transporters in the universal covering group $SU(N_c)$. However, one can also work with parallel transporters taking values in the group $SU(N_c)/\mathbb{Z}(N_c)$. In that case, it is impossible to probe the gluon theory with static test quarks represented by Polyakov loops in the fundamental representation of the gauge group. Instead one is limited to purely gluonic observables. In fact, $SO(3) = SU(2)/\mathbb{Z}(2)$ gauge theories have been studied in detail on the lattice [6, 7, 8, 9, 10, 11, 12]. One finds that lattice artifacts — namely center monopoles — make it difficult to approach the continuum limit in this formulation. There is a phase transition in which the lattice theory sheds off these artifacts, and one then expects it to be equivalent to the standard $SU(2)$ Yang-Mills theory in the continuum limit. This suggests that it is best to formulate lattice gauge theories using the universal covering group, e.g. $SU(N_c)$ rather than $SU(N_c)/\mathbb{Z}(N_c)$, in order to avoid these lattice artifacts. The universal covering group of $SO(N)$ is $Spin(N)$ which also has a non-trivial center: $\mathbb{Z}(2)$ for odd N, $\mathbb{Z}(2) \otimes \mathbb{Z}(2)$ for $N = 4k$, and $\mathbb{Z}(4)$ for $N = 4k + 2$. The center of the group $Sp(N)$ is $\mathbb{Z}(2)$ for all N. Hence, the universal covering groups of all main sequence Lie groups have a non-trivial center. What about the exceptional groups? Interestingly, the groups $G(2)$, $F(4)$, and $E(8)$ have a trivial center and are their own universal covering groups. The groups $E(6)$ and $E(7)$, on the other hand, have the non-trivial centers $\mathbb{Z}(3)$ and $\mathbb{Z}(2)$, respectively. The exceptional Lie group $G(2)$ is the simplest group whose universal covering group has a trivial center.

The triviality of the center has profound consequences for the way in which confinement is realized. In particular, a static "quark" in the fundamental $\{7\}$ representation of $G(2)$ can be screened by three $G(2)$ "gluons" in the adjoint $\{14\}$ representation. As a result, in $G(2)$ Yang-Mills theory the color flux string connecting two static $G(2)$ "quarks" can break due to the creation of dynamical gluons. This phenomenon is reminiscent of full QCD (with an $SU(3)$ color gauge group) in which the string connecting a static quark and anti-quark can break due to the pair creation of light dynamical quarks. Indeed, 6 of the 14 $G(2)$ gluons transform as $\{3\}$ and $\{\overline{3}\}$ under the $SU(3)$ subgroup of $G(2)$ and thus qualitatively behave like dynamical quarks and anti-quarks. In particular, they explicitly break the $\mathbb{Z}(3)$ center symmetry of the $SU(3)$ subgroup down to the trivial center of $G(2)$. The remaining $14 - 6 = 8$ $G(2)$ "gluons" transform as $\{8\}$ under the $SU(3)$ subgroup and hence resemble the ordinary gluons familiar from QCD. It should be pointed out that — despite the broken string — just like full QCD, $G(2)$ Yang-Mills theory is still expected to confine color. In particular, one does not expect colored states of single $G(2)$ "gluons" in the physical spectrum. The triviality of the center of $G(2)$ Yang-Mills theory also affects the physics at high temperatures. In particular, there is not necessarily a deconfinement phase transition, and we expect merely a crossover between a low-temperature "glueball" regime and a high-temperature $G(2)$ "gluon" plasma. Due to the triviality of the center, unlike e.g. for $SU(N_c)$ Yang-Mills theory, there is no qualitative difference between the low- and the high-temperature regimes because the Polyakov loop is no longer a good order parameter.

It is often being asked which degrees of freedom are responsible for confinement. Popular candidates are dense instantons, merons, Abelian monopoles and center vortices. Center vortices (and 't Hooft twist sectors) are absent in $G(2)$ gauge theories, while the other topological objects potentially exist, although their identification is a very subtle issue that often involves unsatisfactory gauge fixing procedures. At strong coupling $G(2)$ lattice gauge theories still confine without a center. Hence, center vortices should not be necessary to explain the absence of colored states in the physical spectrum [13]. Still, the center plays an important role for the finite temperature deconfinement phase transition in $SU(N_c)$ Yang-Mills theory, and center vortices may well be relevant for this physics. If $G(2)$ Yang-Mills theory indeed has no finite temperature deconfinement phase transition, one might argue that this is due to the absence of center vortices and twist sectors. Assuming that they can be properly defined, Abelian monopoles are potentially present in $G(2)$ gauge theory, and might be responsible for the absence of colored states. On the other hand, if — despite of the existence of Abelian monopoles — a deconfinement phase transition does not exist in $G(2)$ Yang-Mills theory, monopoles might not be responsible for the physics of deconfinement. In any case, quantifying these issues in a concrete way is a very difficult task.

The exceptional confinement in $G(2)$ gauge theory can be smoothly connected with the usual $SU(3)$ confinement by exploiting the Higgs mechanism. When a scalar field in the fundamental $\{7\}$ representation of $G(2)$ picks up a vacuum expectation value, the gauge symmetry breaks down to $SU(3)$, and the 6 additional $G(2)$ "gluons" become massive. By progressively increasing the vacuum expectation value of the Higgs field, one can decouple those particles, thus smoothly interpolating between $G(2)$ and $SU(3)$ gauge theories. In this way, we use $G(2)$ gauge theories as a theoretical laboratory in which the $SU(3)$ theories we are most interested in are embedded in an unusual environment. This provides theoretical insight not only into the exceptional $G(2)$ confinement, but also into the $SU(3)$ confinement that occurs in Nature.

The rest of the paper is organized as follows. In section 2 we review the center symmetry, the construction of the Polyakov loop, and some subtle issues related to the physics of non-Abelian gauge fields in a finite volume. Some details of periodic and C-periodic boundary conditions are discussed in an appendix. In section 3 we present the basic features of the exceptional group $G(2)$. Section 4 contains the discussion of various field theories with gauge group $G(2)$. As a starting point, we consider $G(2)$ Yang-Mills theory, which we then break to the $SU(3)$ subgroup using the Higgs mechanism. We then add fermion fields in both the fundamental and the adjoint representation, thus arriving at $G(2)$ QCD and supersymmetric $G(2)$ Yang-Mills theory, respectively. In both cases, we concentrate on the chiral symmetries and we discuss how they are realized at low and at high temperature. In section 5 we substantiate the qualitative pictures painted in section 4 by performing strong coupling calculations in $G(2)$ lattice gauge theory. In particular, we show that the theory confines although there is no string tension. Finally, section 6 contains our conclusions.

2 Center Symmetry, Polyakov Loop and Gauge Fields in a Finite Volume

When defined properly, the Polyakov loop is a useful order parameter in Yang-Mills gauge theories with a non-trivial center symmetry, which distinguishes confinement at low temperatures from deconfinement at high temperatures. In particular, the Polyakov loop varies under non-trivial transformations in the center of the gauge group and it thus signals the spontaneous breakdown of the center symmetry at high temperatures. The expectation value of the Polyakov loop $\langle\Phi\rangle = \exp(-\beta F)$ measures the free energy F of an external static test quark. In a confined phase with unbroken non-trivial center symmetry the free energy of a static quark is infinite. Hence, $\langle\Phi\rangle = 0$ and the center symmetry is unbroken. In a deconfined phase, on the other hand, F is finite, $\langle\Phi\rangle \neq 0$, and the center symmetry is spontaneously broken.

The Polyakov loop is a rather subtle observable whose definition needs special care. In particular, it is sensitive to spatial and temporal boundary conditions. For example, for a system of $SU(3)$ Yang-Mills gluons on a finite torus, the expectation value of the Polyakov loop is always zero even in the deconfined phase [14]. This is a consequence of the $\mathbf{Z}(3)$ Gauss law: a single static test quark cannot exist in a periodic volume because its color flux cannot go to infinity and must thus end in an anti-quark. Due to the Gauss law, a torus is always neutral. Since it always vanishes, on a finite torus the expectation value of the Polyakov loop does not contain any useful information about confinement or deconfinement. Still, using the Polyakov loop, one can, for example, define its finite volume constraint effective potential, which does indeed allow one to distinguish confined from deconfined phases.

Let us consider a non-Abelian Yang-Mills theory with gauge group G and anti-Hermitean vector potential $A_\mu(x)$. The physics is invariant under non-Abelian gauge transformations

$$A_\mu(x)' = \Omega(x)(A_\mu(x) + \partial_\mu)\Omega(x)^\dagger, \tag{2.1}$$

where $\Omega(x) \in G$. We now put the system in a finite 4-dimensional rectangular space-time volume of size $L_1 \times L_2 \times L_3 \times L_4$. Here L_i is the extent in the spatial i-direction and $L_4 = \beta = 1/T$ is the extent of periodic Euclidean time which determines the inverse temperature $\beta = 1/T$. We consider periodic boundary conditions in both space and Euclidean time, such that our 4-dimensional space-time volume is a hyper-torus. This means that gauge-invariant physical quantities — but not the gauge-dependent vector potentials themselves — are periodic functions of space-time. The gauge fields themselves must be periodic only up to gauge transformations, i.e.

$$A_\mu(x + L_\nu e_\nu) = \Omega_\nu(x)(A_\mu(x) + \partial_\mu)\Omega_\nu(x)^\dagger. \tag{2.2}$$

Here e_ν is the unit-vector in the ν-direction and $\Omega_\nu(x)$ is a gauge transformation that relates the gauge field $A_\mu(x+L_\nu e_\nu)$, shifted by a distance L_ν in the ν-direction, to the unshifted gauge field $A_\mu(x)$. Mathematically speaking, the $\Omega_\nu(x)$ define a universal fiber bundle of transition functions which glue the torus together at the boundaries. As explained in the appendix, the transition functions must obey the cocycle condition

$$\Omega_\nu(x + L_\rho e_\rho)\Omega_\rho(x) = \Omega_\rho(x + L_\nu e_\nu)\Omega_\nu(x) z_{\nu\rho}. \tag{2.3}$$

This consistency conditions contains the twist-tensor $z_{\nu\rho}$ which takes values in the center of the gauge group.

It should be noted that the transition functions $\Omega_\nu(x)$ are physical degrees of freedom of the non-Abelian gauge field, just like the vector potentials $A_\mu(x)$ themselves. In particular, in the path integral one must also integrate over the transition functions, otherwise gauge-variant unphysical quantities like $A_\mu(x)$ itself might also become periodic. Under general (not necessarily periodic) gauge transformations $\Omega(x)$ the transition functions transform as

$$\Omega_\nu(x)' = \Omega(x + L_\nu e_\nu)\Omega_\nu(x)\Omega(x)^\dagger. \tag{2.4}$$

In lattice gauge theory the transition functions are nothing but the Wilson parallel transporters on the links that connect two opposite sides of the periodic volume.

The twist-tensor is gauge-invariant. Hence, as was first pointed out by 't Hooft [15], non-Abelian gauge fields on a torus fall into gauge equivalence classes characterized by the twist-tensor, which provides a gauge-invariant characterization of distinct superselection sectors. A non-trivial twist-tensor $z_{\nu\rho} \neq 0$ implies background electric or magnetic fluxes that wrap around the torus in various directions, while the sector with trivial twist $z_{\nu\rho} = 1$ describes a periodic world without electric or magnetic fluxes. It should be noted that one need not sum over the different twist-sectors in the path integral, because they correspond to distinct superselection sectors of the theory.

Interestingly, in a non-Abelian Yang-Mills theory there is a symmetry transformation

$$\Omega_\mu(x)' = \Omega_\mu(x) z_\mu, \tag{2.5}$$

which leaves the boundary condition as well as the twist-tensor — and hence the superselection sector — invariant. Here z_μ is an element of the center of the gauge group G. This center symmetry exists only if all fields in the theory are center-blind. This is automatically the case for the gauge fields which transform in the adjoint representation. However, if there are fields that transform non-trivially under the center, the center symmetry is explicitly broken. For example, a matter field that transforms as

$$\Psi(x)' = \Omega(x)\Psi(x), \tag{2.6}$$

under gauge transformations, obeys the boundary condition

$$\Psi(x + L_\mu e_\mu) = \Omega_\mu(x)\Psi(x), \tag{2.7}$$

which is gauge-covariant, but not invariant under the center symmetry of eq.(2.5).

Based on the previous discussion, we are finally ready to attempt a first definition of the Polyakov loop

$$\Phi(\vec{x}) = \mathrm{Tr}[\Omega_4(\vec{x},0)\mathcal{P}\exp(\int_0^\beta dt\ A_4(\vec{x},t))], \tag{2.8}$$

which is invariant under the transformations of eq.(2.4) only because the transition function $\Omega_4(\vec{x},0)$ is included in its definition. Then one obtains

$$\begin{aligned}
\Phi(\vec{x})' &= \mathrm{Tr}[\Omega_4(\vec{x},0)'\mathcal{P}\exp(\int_0^\beta dt\ A_4(\vec{x},t)')] \\
&= \mathrm{Tr}[\Omega(\vec{x},\beta)\Omega_4(\vec{x},0)\Omega(\vec{x},0)^\dagger\Omega(\vec{x},0)\mathcal{P}\exp(\int_0^\beta dt\ A_4(\vec{x},t))\Omega(\vec{x},\beta)^\dagger] \\
&= \Phi(\vec{x}).
\end{aligned} \tag{2.9}$$

Under the center symmetry transformation of eq.(2.5) the Polyakov loop transforms as

$$\Phi(\vec{x})' = \Phi(\vec{x}) z_4, \tag{2.10}$$

and thus it provides us with an order parameter for the spontaneous breakdown of the center symmetry. However, on a torus the expectation value of the Polyakov loop $\langle\Phi\rangle$ always vanishes, simply because spontaneous symmetry breaking — in the sense of a non-vanishing order parameter — does not occur in a finite volume. Alternatively, one may say that the expectation value of the Polyakov loop always vanishes because the presence of a single static quark is incompatible with the Gauss law on a torus. In any case, since it is always zero, on a finite torus the expectation value of the Polyakov loop does not contain any information about confinement or deconfinement, or about how the center symmetry is dynamically realized.

Still, even on a finite torus the Polyakov loop can be used to define related quantities that indeed contain useful information about confinement versus deconfinement, and about the realization of the center symmetry. For example, one can define the finite volume ($\beta V = L_1 L_2 L_3 L_4$) constraint effective potential $\mathcal{V}(\Phi)$ of the Polyakov loop as

$$\begin{aligned} \exp(-\beta V \mathcal{V}(\Phi)) &= \int \mathcal{D}A\ \delta(\Phi - \frac{1}{V}\int d^3x\ \mathrm{Tr}[\Omega_4(\vec{x},0)\mathcal{P}\exp(\int_0^\beta dt\ A_4(\vec{x},t))]) \\ &\times\ \exp(-S[A]), \end{aligned} \tag{2.11}$$

where $S[A]$ is the Euclidean Yang-Mills action. In the confined phase, the constraint effective potential $\mathcal{V}(\Phi)$ has its minimum at $\Phi = 0$, while in the deconfined phase it has degenerate minima at $\Phi \neq 0$ which are related to one another by center symmetry transformations.

One may still not be satisfied with the previous definition of the Polyakov loop. In particular, one may argue that the center symmetry transformations are part of the gauge group. In that case, the Polyakov loop, as defined in eq.(2.8), would simply be a gauge-variant unphysical quantity. In order to resurrect the Polyakov loop from this deadly argument, we now discuss a space-time volume with C-periodic boundary conditions in the spatial directions. Thermodynamics dictates that the boundary conditions in the Euclidean time direction remain periodic. Even the expectation value of the Polyakov loop itself becomes a useful observable when C-periodic boundary conditions are used in $SU(3)$ Yang-Mills theory. In that case, a spatial shift by a distance L_i is accompanied by a charge-conjugation transformation [16, 17]. A single static quark can exist in a C-periodic box because its color flux can end in a mirror anti-quark on the other side of the boundary. As a consequence, the expectation value of the Polyakov loop no longer vanishes automatically [18]. In fact, it now vanishes only if one takes the infinite volume limit in the confined phase, while it remains non-zero in the deconfined phase.

In a C-periodic volume the physics is periodic up to a charge-conjugation twist, i.e. all physical quantities are replaced by their charge-conjugates when shifted by a distance L_i in a spatial direction. Of course, the gauge fields themselves are C-periodic only up to gauge transformations, i.e.

$$A_\mu(x + L_i e_i) = \Omega_i(x)(A_\mu(x)^* + \partial_\mu)\Omega_i(x)^\dagger. \tag{2.12}$$

Here $A_\mu(x)^*$ is the charge-conjugate of the gauge field $A_\mu(x)$. In the Euclidean time direction we keep periodic boundary conditions, i.e.

$$A_\mu(x + \beta e_i) = \Omega_4(x)(A_\mu(x) + \partial_\mu)\Omega_4(x)^\dagger. \tag{2.13}$$

As shown in the appendix, the cocycle conditions for C-periodic boundary conditions are given by

$$\begin{aligned} &\Omega_i(x + L_j e_j)\Omega_j(x)^* = \Omega_j(x + L_i e_i)\Omega_i(x)^* z_{ij}, \\ &\Omega_i(x + \beta e_4)\Omega_4(x)^* = \Omega_4(x + L_i e_i)\Omega_i(x) z_{i4}, \end{aligned} \tag{2.14}$$

and hence they differ from those for periodic boundary conditions. With C-periodic boundary conditions the transition functions transform under gauge transformations as

$$\Omega_i(x)' = \Omega(x + L_i e_i)\Omega_i(x)\Omega(x)^T, \ \Omega_4(x)' = \Omega(x + \beta e_4)\Omega_4(x)\Omega(x)^\dagger, \tag{2.15}$$

where T denotes the transpose. As we work out in the appendix, unlike for periodic boundary conditions, there are constraints on the twist-tensor itself. First

$$z_{ij}^2 z_{jk}^2 z_{ki}^2 = 1, \tag{2.16}$$

and second

$$z_{i4}^2 = z_{j4}^2. \tag{2.17}$$

Interestingly, with C-periodic boundary conditions the twist-tensor is no longer invariant against the center symmetry transformations of eq.(2.5). One finds

$$z'_{ij} = z_{ij} z_i^2 z_j^{*2}, \ z'_{i4} = z_{i4} z_4^{*2}. \tag{2.18}$$

These relations can be used to relate gauge-equivalent twist-tensors to one another.

First, let us assume that the center of the gauge group G is $\mathbf{Z}(N_c)$ with odd N_c. This is the case for $SU(N_c)$ groups with odd N_c as well as for $E(6)$ which has the center $\mathbf{Z}(3)$. Of course, the physical color gauge group $SU(3)$ with its center $\mathbf{Z}(3)$ also falls in this class. In that case, in a C-periodic volume all twist-sectors are gauge-equivalent. In particular, using eq.(2.17) and putting the transformation parameter to $z_4^2 = z_{i4}$ one obtains $z'_{i4} = 1$ for all i. Next, we put the transformation parameter $z_1 = 1$ and we choose $z_2^2 = z_{12}$ such that $z'_{12} = 1$, and $z_3^2 = z_{13}$ such that $z'_{13} = 1$. Using the consistency condition eq.(2.16) one finally obtains ${z'_{23}}^2 = {z'_{21}}^2 {z'_{13}}^2 = 1$ such that $z'_{23} = 1$. Hence, in this case the entire twist-tensor $z'_{\mu\nu} = 1$ is trivial. Consequently, for gauge groups with the center $\mathbf{Z}(N_c)$ with odd N_c there is only one C-periodic boundary condition. In other words, for C-periodic boundary conditions no analog of 't Hooft's electric and magnetic flux sectors exists — all twist-sectors are gauge-equivalent to the trivial one. In these cases, according to eq.(2.17) the twist-tensor element z_{i4} is independent of the spatial direction i. Furthermore, since

N_c is odd, its square-root in the center $\sqrt{z_{i4}} \in \mathbf{Z}(N_c)$ is uniquely defined (without any sign-ambiguity). According to eq.(2.18) it transforms as

$$\sqrt{z'_{i4}} = \sqrt{z_{i4}} z_4^*, \tag{2.19}$$

under center transformations. This finally allows us to write down a completely gauge-invariant definition of the Polyakov loop in a C-periodic volume

$$\Phi(\vec{x}) = \mathrm{Tr}[\sqrt{z_{i4}} \Omega_4(\vec{x}, 0) \mathcal{P} \exp(\int_0^\beta dt \ A_4(\vec{x}, t))]. \tag{2.20}$$

Unlike for periodic boundary conditions, the center transformation of the transition function $\Omega_4(\vec{x}, 0)' = \Omega_4(\vec{x}, 0) z_4$ is now compensated by the variation of the square-root of the twist-tensor from eq.(2.19) and one obtains $\Phi(\vec{x})' = \Phi(\vec{x})$. Defined in this completely gauge-invariant way, the expectation value of the Polyakov loop $\langle \Phi \rangle = \exp(-\beta F)$ indeed determines the free energy F of a single static quark. In contrast to the periodic torus, a C-periodic volume can contain a single static quark, because the color flux string emanating from it can end in a charge-conjugate anti-quark on the other side of the boundary.

The groups $SU(N_c)$ with even N_c have a sign-ambiguity in the definition of the square-root of a center element. In that case, the expectation value of the Polyakov loop vanishes even in a C-periodic volume. The same is true for $Spin(N)$ — the universal covering group of $SO(N)$ — which has the center $\mathbf{Z}(2)$ for odd N, $\mathbf{Z}(2) \otimes \mathbf{Z}(2)$ for $N = 4k$, and $\mathbf{Z}(4)$ for $N = 4k + 2$, as well as for the symplectic groups $Sp(N)$ and the exceptional group $E(7)$ which both have the center $\mathbf{Z}(2)$. In those cases, one is limited to the construction of the finite volume constraint effective potential. The exceptional groups $G(2)$, $F(4)$ and $E(8)$ have a trivial center. Then the Polyakov loop is not an order parameter, but it can at least be defined without any problems even in a simple periodic volume.

Keeping in mind the subtleties discussed above, when we refer to the Polyakov loop in the rest of this paper, strictly speaking, we mean the location of a minimum of its constraint effective potential on the torus, or its expectation value in a C-periodic box. Both are identical in the infinite volume limit.

3 The Exceptional Group $G(2)$

In this section we discuss some basic properties of the Lie group $G(2)$ — the simplest among the exceptional groups $G(2)$, $F(4)$, $E(6)$, $E(7)$ and $E(8)$ — which do not fit into the main sequences $SO(N) \simeq Spin(N)$, $SU(N)$ and $Sp(N)$. While there is only one non-Abelian compact Lie algebra of rank 1 — namely the one of $SO(3) \simeq SU(2) = Sp(1)$ — there are four of rank 2. These rank 2 algebras generate the groups $G(2)$, $SO(5) \simeq Sp(2)$, $SU(3)$ and $SO(4) \simeq SU(2) \otimes SU(2)$, which have 14, 10, 8 and 6 generators, respectively. For us the group $G(2)$ is of particular interest because it has a trivial center and is its own universal covering group. As we will see later, this has interesting consequences for the confinement mechanism.

It is natural to construct $G(2)$ as a subgroup of $SO(7)$ which has rank 3 and 21 generators. The 7×7 real orthogonal matrices Ω of the group $SO(7)$ have determinant 1 and obey the constraint

$$\Omega_{ab}\Omega_{ac} = \delta_{bc}. \tag{3.1}$$

The $G(2)$ subgroup contains those matrices that, in addition, satisfy the cubic constraint

$$T_{abc} = T_{def}\Omega_{da}\Omega_{eb}\Omega_{fc}. \tag{3.2}$$

Here T is a totally anti-symmetric tensor whose non-zero elements follow by anti-symmetrization from

$$T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1. \tag{3.3}$$

The tensor T also defines the multiplication rules for octonions [19]. Eq.(3.3) implies that eq.(3.2) represents 7 non-trivial constraints which reduce the 21 degrees of freedom of $SO(7)$ to the 14 parameters of $G(2)$. It should be noted that $G(2)$ inherits the reality properties of $SO(7)$: all its representations are real.

We make the following choice for the first 8 generators of $G(2)$ in the 7-dimensional fundamental representation [19]

$$\Lambda_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_a & 0 & 0 \\ 0 & -\lambda_a^* & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

Here λ_a (with $a \in \{1, 2, ..., 8\}$) are the usual 3×3 Gell-Mann generators of $SU(3)$ which indeed is a subgroup of $G(2)$. We have chosen the standard normalization $\mathrm{Tr}\lambda_a\lambda_b = \mathrm{Tr}\Lambda_a\Lambda_b = 2\delta_{ab}$. The representation we have chosen involves complex numbers. However, it is unitarily equivalent to a representation that is entirely real. In the chosen basis of the generators it is manifest that under $SU(3)$ subgroup transformations the 7-dimensional representation decomposes into

$$\{7\} = \{3\} \oplus \{\overline{3}\} \oplus \{1\}. \tag{3.5}$$

Since $G(2)$ has rank 2, only two generators can be diagonalized simultaneously. In our choice of basis these are the $SU(3)$ subgroup generators Λ_3 and Λ_8. Consequently, just as for $SU(3)$, the weight diagrams of $G(2)$ representations can be drawn in a 2-dimensional plane. For example, the weight diagram of the fundamental representation is shown in figure 1. One notes that it is indeed a superposition of the weight diagrams of a $\{3\}$, $\{\overline{3}\}$ and $\{1\}$ in $SU(3)$. Since all $G(2)$ representation are real, the $\{7\}$ representation is equivalent to its complex conjugate. As a

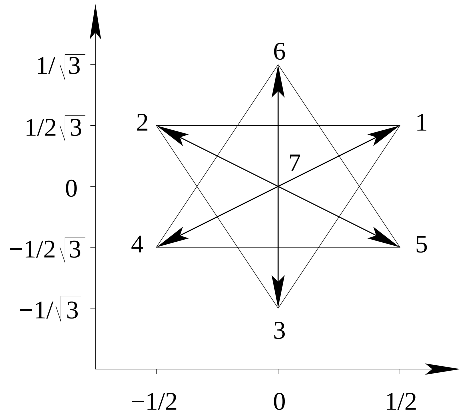

Figure 1: *The weight diagram of the 7-dimensional fundamental representation of* $G(2)$ *(rescaled by a factor* $\sqrt{2}$*).*

consequence, $G(2)$ "quarks" and "anti-quarks" are indistinguishable. In particular, a $G(2)$ "quark" $\{7\}$ consists of an $SU(3)$ quark $\{3\}$, an $SU(3)$ anti-quark $\{\overline{3}\}$ and an $SU(3)$ singlet $\{1\}$. It should be noted that the $\{3\} \oplus \{\overline{3}\}$ contained in the $\{7\}$ of $G(2)$ corresponds to a real reducible 6-dimensional representation of $SU(3)$.

As usual,

$$\begin{aligned}
T^+ &= \frac{1}{\sqrt{2}}(\Lambda_1 + i\Lambda_2) = |1\rangle\langle 2| - |5\rangle\langle 4|, \\
T^- &= \frac{1}{\sqrt{2}}(\Lambda_1 - i\Lambda_2) = |2\rangle\langle 1| - |4\rangle\langle 5|, \\
U^+ &= \frac{1}{\sqrt{2}}(\Lambda_4 + i\Lambda_5) = |2\rangle\langle 3| - |6\rangle\langle 5|, \\
U^- &= \frac{1}{\sqrt{2}}(\Lambda_4 - i\Lambda_5) = |3\rangle\langle 2| - |5\rangle\langle 6|, \\
V^+ &= \frac{1}{\sqrt{2}}(\Lambda_4 + i\Lambda_6) = |1\rangle\langle 3| - |6\rangle\langle 4|, \\
V^- &= \frac{1}{\sqrt{2}}(\Lambda_6 - i\Lambda_4) = |3\rangle\langle 1| - |4\rangle\langle 6|, \qquad (3.6)
\end{aligned}$$

define $SU(3)$ shift operations between the different states $|1\rangle$, $|2\rangle$,...,$|7\rangle$ in the fundamental representation. The remaining 6 generators of $G(2)$ also define shifts

$$\begin{aligned}
X^+ &= \frac{1}{\sqrt{2}}(\Lambda_9 + i\Lambda_{10}) = |2\rangle\langle 4| - |1\rangle\langle 5| - \sqrt{2}|7\rangle\langle 3| - \sqrt{2}|6\rangle\langle 7|, \\
X^- &= \frac{1}{\sqrt{2}}(\Lambda_9 - i\Lambda_{10}) = |4\rangle\langle 2| - |5\rangle\langle 1| - \sqrt{2}|3\rangle\langle 7| - \sqrt{2}|7\rangle\langle 6|, \\
Y^+ &= \frac{1}{\sqrt{2}}(\Lambda_{11} + i\Lambda_{12}) = |6\rangle\langle 1| - |4\rangle\langle 3| - \sqrt{2}|2\rangle\langle 7| - \sqrt{2}|7\rangle\langle 5|,
\end{aligned}$$

$$Y^- = \frac{1}{\sqrt{2}}(\Lambda_{11} - i\Lambda_{12}) = |1\rangle\langle 6| - |3\rangle\langle 4| - \sqrt{2}|7\rangle\langle 2| - \sqrt{2}|5\rangle\langle 7|,$$
$$Z^+ = \frac{1}{\sqrt{2}}(\Lambda_{13} + i\Lambda_{14}) = |3\rangle\langle 5| - |2\rangle\langle 6| - \sqrt{2}|7\rangle\langle 1| - \sqrt{2}|4\rangle\langle 7|,$$
$$Z^- = \frac{1}{\sqrt{2}}(\Lambda_{13} - i\Lambda_{14}) = |5\rangle\langle 3| - |6\rangle\langle 2| - \sqrt{2}|1\rangle\langle 7| - \sqrt{2}|7\rangle\langle 4|. \quad (3.7)$$

The generators themselves transform under the 14-dimensional adjoint representation of $G(2)$ whose weight diagram is shown in figure 2. From this diagram one sees

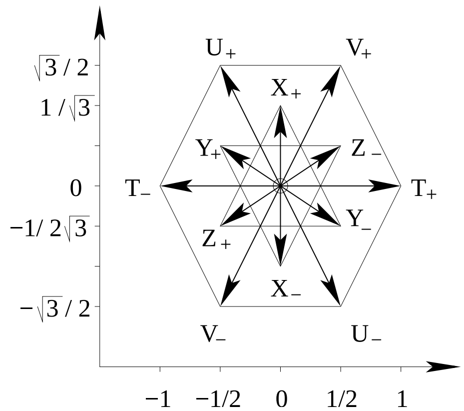

Figure 2: *The weight diagram of the 14-dimensional adjoint representation of* $G(2)$ *(rescaled by a factor* $\sqrt{2}$*).*

that under an $SU(3)$ subgroup transformation the adjoint representation of $G(2)$ decomposes into

$$\{14\} = \{8\} \oplus \{3\} \oplus \{\overline{3}\}. \quad (3.8)$$

This implies that $G(2)$ "gluons" $\{14\}$ consist of the usual $SU(3)$ gluons $\{8\}$ as well as of 6 additional "gluons" with $SU(3)$ quark and anti-quark color quantum numbers $\{3\}$ and $\{\overline{3}\}$.

Let us now discuss the center of $G(2)$. It is interesting to note that the maximal Abelian (Cartan) subgroup of both $G(2)$ and $SU(3)$ is $U(1)^2$ which must contain the center in both cases. Since $G(2)$ contains $SU(3)$ as a subgroup its center cannot be bigger than $\mathbb{Z}(3)$ (the center of $SU(3)$) because the potential center elements of $G(2)$ must commute with all $G(2)$ matrices (not just with the elements of the $SU(3)$ subgroup). In the fundamental representation of $G(2)$ the center elements of the $SU(3)$ subgroup are given by

$$Z = \left(\begin{array}{ccc} z\mathbb{1} & 0 & 0 \\ 0 & z^*\mathbb{1} & 0 \\ 0 & 0 & 1 \end{array}\right). \quad (3.9)$$

where $\mathbb{1}$ is the 3×3 unit matrix and $z \in \{1, \exp(\pm 2\pi i/3\}$ is an element of $\mathbb{Z}(3)$. By construction, the three 7×7 matrices Z commute with the 8 generators of the $SU(3)$ subgroup of $G(2)$. However, an explicit calculation shows that this is not the case for the remaining 6 generators. Consequently, the center of $G(2)$ is trivial and contains only the identity. The above argument applies to any representation of $G(2)$. In other words, the universal covering group of $G(2)$ is $G(2)$ itself and still it has a trivial center. As we will see, this has drastic consequences for confinement. In particular, the string between static $G(2)$ "quarks" can break already in the pure gauge theory through the creation of dynamical "gluons".

In $SU(3)$ the non-trivial center $\mathbb{Z}(3)$ gives rise to the concept of triality. For example, the trivial representation $\{1\}$ and the adjoint representation $\{8\}$ of $SU(3)$ have trivial triality, while the fundamental $\{3\}$ and anti-fundamental $\{\overline{3}\}$ have non-trivial opposite trialities. Since its center is trivial, the concept of triality does not extend to $G(2)$. In particular, as one can see from eqs.(3.5,3.8), $G(2)$ representations decompose into mixtures of $SU(3)$ representations with different trialities. This has interesting consequences for the results of $G(2)$ tensor decompositions [20]. For example, in contrast to the $SU(3)$ case, the product of two fundamental representations

$$\{7\} \otimes \{7\} = \{1\} \oplus \{7\} \oplus \{14\} \oplus \{27\}, \tag{3.10}$$

contains both the trivial and the adjoint representation. The $\{1\}$ and $\{27\}$ representations are symmetric under the exchange of the two $\{7\}$ representations, while $\{7\}$ and $\{14\}$ are anti-symmetric. As a result of eq.(3.10), already two $G(2)$ "quarks" can form a color-singlet. Just as for $SU(3)$, three $G(2)$ "quarks" can form a color-singlet "baryon" because

$$\{7\} \otimes \{7\} \otimes \{7\} = \{1\} \oplus 4\,\{7\} \oplus 2\,\{14\} \oplus 3\,\{27\} \oplus 2\,\{64\} \oplus 3\,\{77\}. \tag{3.11}$$

Another interesting example is

$$\begin{aligned} \{14\} \otimes \{14\} \otimes \{14\} &= \{1\} \oplus \{7\} \oplus 5\,\{14\} \oplus 3\,\{27\} \oplus 2\,\{64\} \oplus 4\,\{77\} \oplus 3\,\{77'\} \\ &\oplus \{182\} \oplus 3\,\{189\} \oplus \{273\} \oplus 2\,\{448\}. \end{aligned} \tag{3.12}$$

As a consequence of the absence of triality, the decomposition of the tensor product of three adjoint representations contains the fundamental representation. This means that three $G(2)$ "gluons" G can screen a single $G(2)$ "quark" q, and thus a color-singlet hybrid $qGGG$ can be formed. Later we will also need the results for further tensor product decompositions, two of which are listed here

$$\begin{aligned} &\{7\} \otimes \{14\} = \{7\} \oplus \{27\} \oplus \{64\}, \\ &\{14\} \otimes \{14\} = \{1\} \oplus \{14\} \oplus \{27\} \oplus \{77\} \oplus \{77'\}. \end{aligned} \tag{3.13}$$

It is also interesting to consider the homotopy groups related to $G(2)$ because this tells us what kind of topological excitations can arise. As for $SU(3)$, the third homotopy group of $G(2)$ is

$$\Pi_3[G(2)] = \mathbb{Z}. \tag{3.14}$$

Hence, there are $G(2)$ instantons of any additive integer topological charge and, consequently, also a θ-vacuum angle. Another homotopy group of interest is

$$\Pi_2[G(2)/U(1)^2] = \Pi_1[U(1)^2] = \mathbb{Z}^2. \tag{3.15}$$

Again, this is just like for $SU(3)$. Physically, this means that 't Hooft-Polyakov monopoles with two kinds of magnetic charge show up when $G(2)$ is broken to its maximal Abelian (Cartan) subgroup $U(1)^2$. For $SU(3)$ with center $\mathbb{Z}(3)$ the homotopy

$$\Pi_1[SU(3)/\mathbb{Z}(3)] = \Pi_0[\mathbb{Z}(3)] = \mathbb{Z}(3), \tag{3.16}$$

implies that the pure gauge theory has non-trivial twist-sectors. Interestingly, in contrast to $SU(3)$, for $G(2)$ which has a trivial center $I = \{\mathbb{1}\}$ the first homotopy group

$$\Pi_1[G(2)/I] = \Pi_0[I] = \{0\} \tag{3.17}$$

is trivial. Hence, even in the pure gauge theory non-trivial twist-sectors do not exit.

4 $G(2)$ Gauge Theories

In this section we discuss various theories with gauge group $G(2)$. We start with pure Yang-Mills theory and then add charged matter fields in various representations. For example, we consider a scalar Higgs field in the fundamental representation which breaks $G(2)$ down to $SU(3)$. By varying the vacuum expectation value of the Higgs field one can interpolate between a $G(2)$ and an $SU(3)$ gauge theory. We also add Majorana "quarks" first in the fundamental $\{7\}$ representation and then also in the adjoint $\{14\}$ representation. The former theory is closely related to $SU(3)$ QCD, while the latter corresponds to $\mathcal{N} = 1$ supersymmetric $G(2)$ Yang-Mills theory.

4.1 $G(2)$ Yang-Mills Theory

Let us first consider the simplest $G(2)$ gauge theory — $G(2)$ Yang-Mills theory. Since $G(2)$ has 14 generators there are 14 "gluons". Under the subgroup $SU(3)$ 8 of them transform as ordinary gluons, i.e. as an $\{8\}$ of $SU(3)$. The remaining 6 $G(2)$ gauge bosons break-up into $\{3\}$ and $\{\overline{3}\}$, i.e. they have the color quantum numbers of ordinary quarks and anti-quarks. Of course, in contrast to real quarks, these objects are bosons with spin 1. Still, these additional 6 "gluons" have somewhat similar effects as quarks in full QCD. In particular, they explicitly break the $\mathbb{Z}(3)$ center symmetry of $SU(3)$ and make the center symmetry of $G(2)$ Yang-Mills theory trivial. The Lagrangian for $G(2)$ Yang-Mills theory takes the standard form

$$\mathcal{L}_{YM}[A] = \frac{1}{2g^2}\mathrm{Tr} F_{\mu\nu}F_{\mu\nu}, \tag{4.1}$$

where the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{4.2}$$

is derived from the vector potential

$$A_\mu(x) = igA_\mu^a(x)\Lambda_a. \tag{4.3}$$

The Lagrangian is invariant under non-Abelian gauge transformations

$$A'_\mu = \Omega(A_\mu + \partial_\mu)\Omega^\dagger, \tag{4.4}$$

where $\Omega(x) \in G(2)$. Like all non-Abelian pure gauge theories, $G(2)$ Yang-Mills theory is asymptotically free. Complementary to this, at low energies one expects confinement.

However, in contrast to $SU(3)$ Yang-Mills theory, the triviality of the $G(2)$ center has far reaching consequences for how confinement is realized. In particular, as we have already seen in eq.(3.12), an external static "quark" in the fundamental $\{7\}$ representation can be screened by at least three "gluons". Hence, via creation of dynamical "gluons" the confining string connecting two static $G(2)$ "quarks" can break and the potential flattens off. Hence, the string tension — as the ultimate slope of the heavy "quark" potential at distance $R \to \infty$ — vanishes. Thus, in contrast to $SU(3)$ Yang-Mills theory where gluons cannot screen quarks, there is a more subtle form of confinement in $G(2)$ Yang-Mills theory, very much like the confinement in $SU(3)$ QCD. In the QCD case screening arises due to dynamical quark-anti-quark pair creation, and again the confining string breaks. Thus, $G(2)$ pure Yang-Mills theory provides us with a suitable theoretical laboratory to investigate confinement without facing additional complications due to dynamical fermions. In the next section the issue of $G(2)$ string breaking will be studied in the strong coupling limit of lattice gauge theory.

Of course, unless one proves confinement analytically, one cannot be sure that QCD, $G(2)$ Yang-Mills theory, or any other gauge theory is indeed in the confined phase in the continuum limit. Based on general wisdom, one would certainly expect that $G(2)$ gauge theory confines color in the same way as QCD. In particular, we do not expect it to be in a massless non-Abelian Coulomb phase. An order parameter that distinguishes between a confined phase (without a string tension, however, with color screening) and a Coulomb phase has been constructed by Fredenhagen and Marcu [21]. In the next section we will show that $G(2)$ lattice Yang-Mills theory is indeed in the confined phase in the strong coupling limit.

Due to the triviality of the center one also expects unusual behavior of $G(2)$ Yang-Mills theory at finite temperature. In $SU(N_c)$ Yang-Mills theory there is a deconfinement phase transition at finite temperature where the $\mathbf{Z}(N_c)$ center symmetry gets spontaneously broken. For two colors ($N_c = 2$) the deconfinement phase transition is second order [22, 23, 24, 25, 26] and belongs to the universality class of the 3-d Ising model [27, 28]. For $N_c = 3$, on the other hand, the phase transition is first order [29, 30, 31, 32, 33, 34, 35] and the bulk physics is not universal. This is consistent with what one expects based on the behavior of the 3-d 3-state Potts model [36, 37, 38, 39, 40]. The high-temperature deconfined phase of $SU(N_c)$ Yang-Mills theory is characterized by a non-zero value of the Polyakov loop order parameter and by a vanishing string tension. On the other hand, in the low-temperature confined phase, the Polyakov loop vanishes and the string tension is non-zero. As we have seen before, already in the confined phase of $G(2)$ Yang-Mills theory the string tension is zero. Since for $G(2)$ the center is trivial, the Polyakov loop no longer vanishes in the confined phase and it is hence no longer an order parameter for deconfinement. As a result, for $G(2)$ there is no compelling argument for a phase transition at finite temperature. In particular, a second order phase transition is practically excluded due to the absence of a symmetry that could break spontaneously. Even without an underlying symmetry, a second order phase transition can occur as an endpoint of a line of first order transitions. However, these particular cases require fine-tuning of some parameter and can thus be practically excluded in $G(2)$ Yang-Mills theory. On the other hand, one cannot rule out a first order phase transition because this does not require spontaneous symmetry breaking. Since the deconfinement phase transition in $SU(3)$ Yang-Mills theory is already rather weakly first order, we expect the $G(2)$ Yang-Mills theory to have only a crossover from a low- to a high-temperature regime. In QCD dynamical quarks also break the $\mathbf{Z}(3)$ center symmetry explicitly. As the quark masses are decreased starting from infinity, the first order phase transition of the pure gauge theory persists until it terminates at a critical point and then turns into a crossover [41, 42]. Of course, in the chiral limit there is an exact chiral symmetry that is spontaneously broken at low and restored at high temperatures. Hence, one expects a finite temperature chiral phase transition which should be second order for two and first order for three massless flavors [43]. A second order chiral phase transition will be washed out to a crossover once non-zero quark masses are included. In $G(2)$ Yang-Mills theory there is no chiral symmetry that could provide us with an order parameter for a finite temperature phase transition.

4.2 $G(2)$ Gauge-Higgs Model

In the next step we add a Higgs field in the fundamental $\{7\}$ representation in order to break $G(2)$ spontaneously down to $SU(3)$. Then 6 of the 14 $G(2)$ "gluons" pick up a mass proportional to the vacuum value v of the Higgs field, while the remaining 8 $SU(3)$ gluons are unaffected by the Higgs mechanism and are confined inside glueballs. For large v the theory thus reduces to $SU(3)$ Yang-Mills theory. For small v (on the order of Λ_{QCD}), on the other hand, the additional $G(2)$ "gluons" are light and participate in the dynamics. Finally, for $v = 0$ the Higgs mechanism disappears and we arrive at $G(2)$ Yang-Mills theory. Hence, by varying v one can interpolate smoothly between $G(2)$ and $SU(3)$ Yang-Mills theory and connect the exceptional $G(2)$ confinement with the usual confinement in $SU(3)$.

The Lagrangian of the $G(2)$ gauge-Higgs model is given by

$$\mathcal{L}_{GH}[A,\Phi] = \mathcal{L}_{YM}[A] + \frac{1}{2} D_\mu \Phi D_\mu \Phi + V(\Phi). \tag{4.5}$$

Here $\Phi(x) = (\Phi^1(x), \Phi^2(x), ..., \Phi^7(x))$ is the real-valued Higgs field,

$$D_\mu \Phi = (\partial_\mu + A_\mu)\Phi, \tag{4.6}$$

is the covariant derivative and

$$V(\Phi) = \lambda(\Phi^2 - v^2)^2 \tag{4.7}$$

is the scalar potential. We have seen in eq.(3.11) that the tensor product $\{7\} \otimes \{7\} \otimes \{7\}$ contains a singlet. Hence, one might also expect a cubic term $T_{abc}\Phi^a\Phi^b\Phi^c$ in the Lagrangian. However, due to the anti-symmetry of the tensor T such a term vanishes. The product $\{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\}$ contains four singlets. One corresponds to $v^2\Phi^2$ and one to Φ^4. The other two again vanish due to antisymmetry. Hence, the scalar potential from above is the most general one consistent with $G(2)$ symmetry and perturbative renormalizability.

Let us first consider the ungauged Higgs model with the Lagrangian

$$\mathcal{L}_H[\Phi] = \frac{1}{2} \partial_\mu \Phi \partial_\mu \Phi + V(\Phi). \tag{4.8}$$

This theory has even an enlarged global $SO(7)$ symmetry which is spontaneously broken to $SO(6)$. Due to Goldstone's theorem there are $21 - 15 = 6$ massless bosons and one Higgs particle of mass squared $M_H^2 = 8\lambda v^2$. When we now gauge only the $G(2)$ subgroup of $SO(7)$ we break the global $SO(7)$ symmetry explicitly. As a result, the previously intact global $SO(6) \simeq SU(4)$ symmetry turns into a local $SU(3)$ symmetry. Hence, a Higgs in the $\{7\}$ representation of $G(2)$ breaks the gauge symmetry down to $SU(3)$. The 6 massless Goldstone bosons are eaten and become the longitudinal components of $G(2)$ "gluons" which pick up a mass $M_G = gv$. The remaining 8 gluons are those familiar from $SU(3)$ Yang-Mills theory. Choosing the vacuum value of the Higgs field as $\Phi(x) = (0, 0, 0, 0, 0, 0, v)$, the unbroken $SU(3)$ invariance can be explicitly verified using eqs.(3.4,3.7).

It is interesting to compare this situation with what happens in the standard model. Before gauge interactions are switched on, the standard model Higgs field can be viewed as a vector in the $\{4\}$ representation of $SO(4) \simeq SU(2)_L \otimes SU(2)_R$. When it picks up a vacuum expectation value this global symmetry is spontaneously broken to $SO(3) \simeq SU(2)_{L=R}$ and there are $6 - 3 = 3$ massless Goldstone bosons. When one gauges only the $SU(2)_L \otimes U(1)_Y$ subgroup of $SU(2)_L \otimes SU(2)_R$ one again breaks the global $SO(4)$ symmetry explicitly. As a result, the previously intact global $SO(3) \simeq SU(2)_{L=R}$ symmetry turns into the local $U(1)_{L=R} = U(1)_{em}$ symmetry of electromagnetism. In this case, the 3 Goldstone bosons, of course, become the longitudinal components of the massive bosons $W^{\pm}$ and Z^0.

With the Higgs mechanism in place, we can think of the $G(2)$ model from above as an $SU(3)$ gauge theory with 6 additional vector bosons of mass M_G in the $\{3\}$ and $\{\overline{3}\}$ representation and a scalar Higgs boson with mass M_H as a $\{1\}$ of $SU(3)$. Just like dynamical quarks in QCD, the massive "gluons" in the $\{3\}$ and $\{\overline{3}\}$ representation explicitly break the center of $SU(3)$. As a result, the confining string connecting a static quark-anti-quark pair can break by the creation of massive $G(2)$ "gluons". As the mass of these "gluons" increases with v, the distance at which the string breaks becomes larger. In the limit $v \to \infty$ the additional "gluons" are removed from the theory, the $\mathbb{Z}(3)$ center symmetry is restored, and the unbreakable string of $SU(3)$ Yang-Mills theory emerges.

Using the Higgs mechanism to interpolate between $SU(3)$ and $G(2)$ Yang-Mills theory, we again consider the issue of the deconfinement phase transition. In the $SU(3)$ theory this transition is weakly first order. As the mass of the 6 additional $G(2)$ gluons is decreased, the $\mathbb{Z}(3)$ center symmetry of $SU(3)$ is explicitly broken and the phase transition is weakened. Qualitatively, we expect the heavy "gluons" to play a similar role as heavy quarks in $SU(3)$ QCD. Hence, we expect the first order deconfinement phase transition line to end at a critical endpoint before the additional $G(2)$ "gluons" have become massless [41, 42]. In that case, the pure $G(2)$ Yang-Mills theory should have no deconfinement phase transition, but merely a crossover.

4.3 $G(2)$ QCD

Let us now consider $G(2)$ gauge theory with N_f flavors of fermions. As before, we will use the Higgs mechanism induced by a scalar field in the $\{7\}$ representation to interpolate between $G(2)$ and $SU(3)$ QCD. We introduce $G(2)$ "quarks" as Majorana fermions in the fundamental representation. Since all $G(2)$ representations are real, a Dirac fermion simply represents a pair of Majorana fermions. Hence, it is most natural to work with Majorana "quarks" as fundamental objects. Under $SU(3)$ subgroup transformations a $\{7\}$ of $G(2)$ decomposes into $\{3\} \oplus \{\overline{3}\} \oplus \{1\}$. Hence, when $G(2)$ is broken down to $SU(3)$, a $\{7\}$ Majorana "quark" of $G(2)$ turns into an ordinary Dirac quark $\{3\}$ and its anti-quark $\{\overline{3}\}$ as well as a color singlet Majorana fermion that does not participate in the strong interactions. The $G(2)$ Majorana "quark" spinor can be written as

$$\lambda = \begin{pmatrix} \Psi \\ C\overline{\Psi}^T \\ \chi \end{pmatrix}, \tag{4.9}$$

where Ψ is an $SU(3)$ Dirac quark spinor, χ is the color singlet Majorana fermion, and C is the charge-conjugation matrix. The "anti-quark" spinor $\overline{\lambda}$ is related to λ by charge-conjugation

$$\overline{\lambda} = (\overline{\Psi}, -\Psi^T C^{-1}, -\chi^T C^{-1}). \tag{4.10}$$

The Lagrangian of $G(2)$ QCD takes the form

$$\mathcal{L}_{QCD}[A, \overline{\lambda}, \lambda] = \mathcal{L}_{YM}[A] + \frac{1}{2}\overline{\lambda}\gamma_\mu(\partial_\mu + A_\mu)\lambda. \tag{4.11}$$

In $G(2)$ gauge theory, quark masses arise from Yukawa couplings to the scalar field as well as from Majorana mass terms. For simplicity, in what follows we consider massless "quarks" only.

4.3.1 The $N_f = 1$ Case

As a first step we consider a single flavor — say the u-quark. Ordinary $N_f = 1$ $SU(3)$ QCD has a $U(1)_B$ symmetry — just baryon number which is unbroken. In particular, there are no massless Goldstone bosons and the theory has a mass-gap. Color singlet states include $u\overline{u}$ mesons with a valance quark and anti-quark as well as a uuu baryon Δ^{++} with three valance quarks. The lightest particle in this theory is presumably a vector-meson similar to the physical ω-meson. Like in ordinary QCD, the pseudo-scalar meson η' gets its mass via the anomaly from topological charge fluctuations. Only in the large N_c limit it becomes a Goldstone boson. For $N_c = 3$ it may or may not be lighter than the vector-meson.

As we have seen before, in $G(2)$ gauge theory "quarks" and "anti-quarks" are indistinguishable. Consequently, the $U(1)_{L=R} = U(1)_B$ baryon number symmetry of $SU(3)$ QCD is reduced to a $\mathbf{Z}(2)_B$ symmetry. One can only distinguish between states with an even and odd number of "quark" constituents. In particular, eq.(3.12) implies that one can construct a colorless state $uGGG$ with one $G(2)$ "quark" screened by three $G(2)$ "gluons". This state mixes with other states containing an odd number of quarks — e.g. with the usual uuu states — to form the $G(2)$ "baryon". In contrast to $SU(3)$ QCD, two $G(2)$ "baryons" (which are odd under $\mathbf{Z}(2)_B$) can annihilate into "mesons". When one uses the Higgs mechanism to break $G(2)$ to $SU(3)$, one can remove the 6 additional $G(2)$ "gluons" by increasing the vacuum value v. As a consequence, the states $uGGG$ become heavy and can no longer mix with uuu. As a result, the standard $U(1)_B$ baryon number symmetry of $SU(3)$ QCD emerges as an approximate symmetry. As long as v remains finite, the heavy $G(2)$ "gluons" mediate weak baryon number violating processes. Only in the limit $v \to \infty$ $U(1)_B$ becomes an exact symmetry.

4.3.2 The $N_f \geq 2$ Case

Let us first remind ourselves of standard two flavor QCD with $SU(3)$ color gauge group. The chiral symmetry then is $SU(2)_L \otimes SU(2)_R \otimes U(1)_B$ which is spontaneously broken to $SU(2)_{L=R} \otimes U(1)_B$. Consequently, there are $7 - 4 = 3$ massless Goldstone pions — π^+, π^0 and π^-. For a $G(2)$ Majorana "quark" left- and right-handed components cannot be rotated independently by unitary transformations $L \in SU(2)_L$ and $R \in SU(2)_R$. In fact, the Majorana condition requires $L = R^*$. Hence, the chiral symmetry of $N_f = 2$ $G(2)$ QCD is $SU(2)_{L=R^*} \otimes \mathbf{Z}(2)_B$. Note that in the same way $U(1)_B = U(1)_{L=R}$ is reduced to $U(1)_{L=R^*=R} = \mathbf{Z}(2)_B$. The reduced chiral symmetry of $G(2)$ QCD is expected to still break spontaneously to the maximal vector subgroup which is now $SU(2)_{L=R^*=R} \otimes \mathbf{Z}(2)_B = SO(2)_{L=R} \otimes \mathbf{Z}(2)_B$. In this case, there are only $3 - 1 = 2$ Goldstone bosons. They can be identified as π^0 and the linear combination of π^+ and π^- that is even under charge-conjugation. The mixing between π^+ and π^- is induced by the exchange of one of the 6 $G(2)$ "gluons" that do not belong to $SU(3)$. The linear combination of π^+ and π^- that is odd under charge-conjugation has a non-zero mass and is not a Goldstone boson of $G(2)$ QCD. When we remove the 6 additional "gluons" via the Higgs mechanism, the mixing of π^+ and π^- becomes weaker as v increases. Consequently, the mass splitting between the charge-conjugation even and odd states also decreases until it ultimately vanishes at $v = \infty$. In this limit the larger chiral symmetry of $SU(3)$ QCD emerges and we are left with three massless pions.

It is straightforward to generalize the above discussion to arbitrary $N_f \geq 2$. For $SU(3)$ QCD with general N_f the chiral symmetry is $SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B$ which is spontaneously broken to $SU(N_f)_{L=R} \otimes U(1)_B$, and there are $N_f^2 - 1$ massless Goldstone bosons. As before, the Majorana condition requires $L = R^*$. Hence, the chiral symmetry of $G(2)$ QCD with N_f massless Majorana "quarks" is $SU(N_f)_{L=R^*} \otimes \mathbf{Z}(2)_B$, which is expected to break spontaneously to $SU(N_f)_{L=R^*=R} \otimes \mathbf{Z}(2)_B = SO(N_f)_{L=R} \otimes \mathbf{Z}(2)_B$. Then there are only $N_f(N_f + 1)/2 - 1$ Goldstone bosons. These consist of $N_f - 1$ neutral Goldstone bosons (π^0 and η for $N_f = 3$) and $N_f(N_f - 1)/2$ charge-conjugation even combinations of charged states ($\pi^+ + \pi^-$, $K^+ + K^-$, $K^0 + \overline{K^0}$ for $N_f = 3$).

4.4 Supersymmetric $G(2)$ Yang-Mills Theory

In this section we add a single flavor adjoint Majorana gluino λ to the pure gluon Yang-Mills theory and thus turn it into $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. First, we compare the $SU(3)$ to the $G(2)$ case.

Let us first describe the situation in $SU(3)$ supersymmetric Yang-Mills theory. Then, in addition to the gluons, there is a color octet of Majorana gluinos. The chiral symmetry of this theory is $\mathbf{Z}(3)_\chi \otimes \mathbf{Z}(2)_B$ where $\mathbf{Z}(2)_B$ is the fermion number symmetry of the Majorana fermion, and $\mathbb{Z}(3)_\chi$ is a remnant of the axial $U(1)_R$ symmetry which is broken by the anomaly. At low temperatures the $\mathbb{Z}(3)_\chi$ symmetry is spontaneously broken through the dynamical generation of a gluino condensate $\langle\lambda\lambda\rangle$. Since both gluons and gluinos are in the adjoint representation, the $\mathbb{Z}(3)$ center of the color gauge group is an exact symmetry of supersymmetric $SU(3)$ Yang-Mills theory. At low temperature this discrete symmetry is unbroken, just as in the non-supersymmetric case. As a result, external static quarks and anti-quarks are confined to one another through an unbreakable color flux string.

As a consequence of the spontaneous breakdown of the discrete $\mathbb{Z}(3)_\chi$ symmetry, there are, in fact, three different low-temperature confined phases which are distinguished by the $\mathbb{Z}(3)_\chi$ phase of the gluino condensate. When such phases coexist with one another, they are separated by confined-confined domain walls with a non-zero interface tension. These walls are topological defects which are characterized by the zeroth homotopy group $\Pi_0[\mathbb{Z}(3)_\chi] = \mathbb{Z}(3)$. Based on M-theory, Witten has argued that the walls behave like D-branes and the confining string (which behaves like a fundamental string) can end on the walls [44]. In a field theoretical context this phenomenon has been explained in [45]. Just like other topological excitations, such as monopoles, cosmic strings and vortices, a supersymmetric domain wall has the unbroken symmetry phase in its core. Consequently, inside the wall (as well as inside the string) the chiral $\mathbb{Z}(3)_\chi$ symmetry is restored. Interestingly, the restoration of $\mathbb{Z}(3)_\chi$ induces the spontaneous breakdown of the center symmetry $\mathbb{Z}(3)$. Hence, the core of a supersymmetric domain wall is in the deconfined phase. When a confining string enters the wall the color flux that it carries spreads out and the string ends.

As one increases the temperature, a transition to a deconfined phase with restored chiral symmetry occurs. As usual, in the deconfined phase the $\mathbb{Z}(3)$ center symmetry is spontaneously broken. When a confined-confined domain wall is heated up to the phase transition, the deconfined phase in its core expands and forms a complete wetting layer whose width diverges at the phase transition [45]. Due to the broken $\mathbb{Z}(3)$ center symmetry there are also three distinct deconfined phases. When those coexist, they are separated by deconfined-deconfined domain walls. As the phase transition is approached from above, similarly, a deconfined-deconfined domain wall splits into a pair of confined-deconfined interfaces and its core turns into a complete wetting layer of confined phase [46, 47].

Let us now ask how the situation is modified for $G(2)$ supersymmetric Yang-Mills theory for which the center is trivial. Interestingly, the remnant chiral symmetry is enhanced to $\mathbb{Z}(4)_\chi \otimes \mathbb{Z}(2)_B$ which breaks spontaneously to $\mathbb{Z}(2)_B$. As a result, similar to the $SU(3)$ case, there are now four different low-temperature chirally broken phases which are characterized by the $\mathbb{Z}(4)_\chi$ phase $\pm 1, \pm i$ of the "gluino" condensate. Due to the triviality of the center, we have the exceptional confinement with a breakable string that we have already discussed in the non-supersymmetric $G(2)$ Yang-Mills theory. When two distinct chirally broken phases coexist, they are again separated by a domain wall. In contrast to the $SU(3)$ case, $G(2)$ strings can not only end on such walls because they can break and thus end anywhere.

When $G(2)$ "gluons" and "gluinos" are heated up, their chiral symmetry is restored in a finite temperature phase transition. In contrast to $G(2)$ pure Yang-Mills theory, a phase transition must exist because there is now an exact spontaneously broken $\mathbb{Z}(4)_\chi$ chiral symmetry for which the "gluino" condensate provides us with an order parameter. As another consequence of the triviality of the center, there is only one high-temperature chirally symmetric phase. In particular, deconfined-deconfined domain walls do not exist. However, there are now two types of domain walls in the low-temperature phase. A wall of type I separates chirally broken phases whose "gluino" condensates are related by a $\mathbb{Z}(4)_\chi$ transformation -1, while for a wall of type II the phases are related by a $\pm i$-rotation. The chiral phase transition may be first or second order. In the latter case the low- and high-temperature phases do not coexist at the phase transition and complete wetting does not arise. However, we find it more natural to expect a first order phase transition. For example, the deconfinement phase transition of an ordinary $SU(4)$ Yang-Mills theory, which has a $\mathbb{Z}(4)$ center symmetry, is first order [48, 49, 50, 51, 52]. If the chiral phase transition of $G(2)$ supersymmetric Yang-Mills theory is first order as well, the low- and high-temperature phases coexist and complete wetting may arise. When a wall of type I is heated up to the phase transition, we expect it to split into a pair of interfaces with a complete wetting layer of chirally symmetric phase in between. It is less clear if complete wetting would also occur for domain walls of type II. We do not enter this discussion here. In any case, complete wetting is no longer needed for strings to end on the walls.

5 $G(2)$ Lattice Gauge Theory at Strong Coupling

In order to substantiate some of the claims made in the previous sections we now formulate $G(2)$ Yang-Mills theory on the lattice and derive some analytic results in the strong coupling limit. As usual, such results do not directly apply to the continuum limit and should ultimately be extended by Monte Carlo simulations into the weak coupling continuum regime. Still, assuming that there is no phase transition separating the strong from the weak coupling regime, the strong coupling results provide insight into dynamical behavior — such as confinement — which persists in the continuum limit. For example, for $G(2)$ — in agreement with the expectations — the lattice strong coupling expansion confirms that the color flux string can break by dynamical "gluon" creation. In this sense, the string tension is zero and the Wilson loop is no longer a good order parameter. In order to characterize the phase of the theory, in particular, in order to distinguish between a massive confinement phase like in full QCD and a massless Coulomb phase, one can use the Fredenhagen-Marcu order parameter [21] which we calculate analytically at strong coupling. Indeed, this confirms that $G(2)$ lattice Yang-Mills theory confines color in the same way as $SU(3)$ QCD.

The construction of $G(2)$ Yang-Mills theory on the lattice follows the standard procedure. The link matrices $U_{x,\mu} \in G(2)$ are group elements in the fundamental $\{7\}$ representation, i.e. they can be chosen entirely real. The standard Wilson plaquette action takes the usual form

$$S[U] = -\frac{1}{g^2}\sum_{\Box} \mathrm{Tr}\ U_{\Box} = -\frac{1}{g^2}\sum_{x,\mu<\nu} \mathrm{Tr}\ U_{x,\mu}U_{x+\hat{\mu},\nu}U^{\dagger}_{x+\hat{\nu},\mu}U^{\dagger}_{x,\nu}, \tag{5.1}$$

where g is the bare gauge coupling. The partition function is given by

$$Z = \int \mathcal{D}U \exp(-S[U]), \tag{5.2}$$

where the measure of the path integral

$$\int \mathcal{D}U = \prod_{x,\mu}\int_{G(2)} dU_{x,\mu}, \tag{5.3}$$

is a product of local Haar measures of the group $G(2)$ for each link. By construction, both the action and the measure are explicitly invariant under gauge transformations

$$U'_{x,\mu} = \Omega_x U_{x,\mu}\Omega^{\dagger}_{x+\hat{\mu}}, \tag{5.4}$$

with $\Omega_x \in G(2)$. The Wilson loop

$$W_{\mathcal{C}} = \mathrm{Tr}\ \mathcal{P}\prod_{(x,\mu)\in\mathcal{C}} U_{x,\mu} \tag{5.5}$$

is the trace of a path ordered product of link variables along the closed path $\mathcal{C}$, and its expectation value is given by

$$\langle W_{\mathcal{C}}\rangle = \frac{1}{Z}\int \mathcal{D}U\ W_{\mathcal{C}} \exp(-S[U]). \tag{5.6}$$

For a rectangular path $\mathcal{C}$ of extent R in the spatial and T in the temporal direction the Wilson loop

$$\lim_{T\to\infty}\langle W_{\mathcal{C}}\rangle = \exp(-V(R)T) \tag{5.7}$$

determines the potential $V(R)$ between static color sources at distance R. In a phase with a linearly rising confining potential $V(R) \sim \sigma R$, where σ is the string tension, the Wilson loop obeys an area law. If the potential levels off at large distances, the Wilson loop obeys a perimeter law. In the strong coupling limit of $SU(3)$ Yang-Mills theory the Wilson loop indeed follows an area law. In $G(2)$ Yang-Mills theory, on the other hand, we expect static "quarks" to be screened by "gluons" and hence the string to break. As a result, the static "quark" potential levels off and large Wilson loops obey a perimeter law.

In the strong coupling regime $g^2 \gg 1$, and $1/g^2$ can be used as a small expansion parameter. The first step of the strong coupling expansion is the character expansion of the Boltzmann factor for an individual plaquette

$$\exp(\frac{1}{g^2}\mathrm{Tr}\ U_\Box) = \sum_\Gamma c_\Gamma(\frac{1}{g^2})\chi_\Gamma(U_\Box), \tag{5.8}$$

where Γ is a generic representation of the gauge group and the corresponding character $\chi_\Gamma(U_\Box)$ is the trace of the matrix $U_\Box$ in that representation. The coefficients $c_\Gamma(1/g^2)$ enter the strong coupling expansion as power-series in $1/g^2$. For example, for $G(2)$ we have

$$\begin{aligned} c_{\{1\}}(\frac{1}{g^2}) &= 1 + \frac{1}{2g^4} + \frac{1}{6g^6} + \frac{1}{6g^8} + \frac{1}{12g^{10}} + \frac{7}{144g^{12}} + ..., \\ c_{\{7\}}(\frac{1}{g^2}) &= \frac{1}{g^2} + \frac{1}{2g^4} + \frac{2}{3g^6} + \frac{5}{12g^8} + \frac{7}{24g^{10}} + \frac{1}{6g^{12}} + ..., \\ c_{\{14\}}(\frac{1}{g^2}) &= \frac{1}{2g^4} + \frac{1}{3g^6} + \frac{3}{8g^8} + \frac{1}{4g^{10}} + \frac{1}{6g^{12}} + ... \end{aligned} \tag{5.9}$$

Let us now compute the expectation value of a rectangular Wilson loop of size $R \times T$ in the strong coupling limit. In $SU(N_c)$ lattice Yang-Mills theory the leading contribution in the strong coupling expansion results from tiling the rectangular surface enclosed by the Wilson loop with $R \times T$ elementary plaquettes in the fundamental representation. This gives rise to the strong coupling area law. All higher order contributions amount to deformations of this minimal surface of plaquettes. Such contributions are also present for $G(2)$. However, in the $G(2)$ case there are additional terms arising from a tube of plaquettes along the perimeter of the Wilson loop. In fact, these contributions dominate at large R and give rise to a strong coupling perimeter law. For small R, on the other hand, the surface term dominates and yields a linearly rising potential at short distances. It should be noted that tube contributions arise even in $SU(N_c)$ Yang-Mills theory for Wilson loops of adjoint charges. In that case, there is again no linearly rising confinement potential. Due to the triviality of the center, for $G(2)$ the perimeter law arises already for fundamental charges.

Here we consider only the leading contribution to the strong coupling expansion. For small $R \leq R_c$ the surface term dominates and gives

$$\langle W_\mathcal{C} \rangle = 7(\frac{1}{7g^2})^{RT} \stackrel{T\to\infty}{\longrightarrow} \exp(-V(R)T), \tag{5.10}$$

which yields a linear potential

$$V(R) = -\log(\frac{1}{7g^2})R. \tag{5.11}$$

In an $SU(N_c)$ Yang-Mills theory the linear potential would extend to arbitrary distances and $-\log(1/7g^2)$ would play the role of the string tension. In the $G(2)$ case, however, the large $R \geq R_c$ behavior is dominated by the perimeter term

$$\langle W_{\mathcal{C}} \rangle = 4(\frac{1}{7g^2})^{8(R+T-2)}, \tag{5.12}$$

which gives rise to a flat (R-independent) potential

$$V(R) = -8\log(\frac{1}{7g^2}). \tag{5.13}$$

At distances larger than $R_c = 8$ the perimeter term overwhelms the surface term and the string breaks. Hence, there is no confinement in the sense of a non-vanishing string tension characterizing the slope of the potential at asymptotic distances. When one pulls apart a $G(2)$ "quark" pair beyond the distance R_c, "gluons" pop up from the vacuum and screen the fundamental color charges of the "quarks". This is possible only because $G(2)$ has a trivial center. From the tensor product decomposition of eq.(3.12) one infers that at least three "gluons" (which are in the $\{14\}$ of $G(2)$) are needed to screen a single "quark" (in the $\{7\}$ of $G(2)$).

Since its string can break, pure $G(2)$ Yang-Mills theory resembles full QCD. In that case, dynamical quark-anti-quark pairs materialize from the vacuum to screen an external static quark-anti-quark pair at large separation. Hence, also in full QCD the static quark-anti-quark potential flattens off and ultimately there is no string tension. Of course, this does not mean that QCD does not confine. In particular, there should be no single quark or gluon states in the physical spectrum, i.e. QCD should not be realized in a non-Abelian Coulomb phase. An order parameter that distinguishes between Coulomb and confinement phases (even if there is no string tension) has been constructed by Fredenhagen and Marcu [21]. This order parameter can also be adapted to $G(2)$ pure Yang-Mills theory and can, in fact, be evaluated in the strong coupling limit.

The Fredenhagen-Marcu order parameter is a ratio of two expectation values. The numerator consists of parallel transporters along an open staple-shaped path connecting a source and a sink of color flux and the denominator is the square root of a closed Wilson loop

$$\rho(R,T) = \frac{\langle \diamond\!\sqcap\!\diamond \rangle}{\langle \square \rangle^{1/2}}. \tag{5.14}$$

The open path symbolic object in the numerator stands for

$$\diamond\!\sqcap\!\diamond = \mathrm{Tr}(U_{\Box_x}\Lambda_a)\mathrm{Tr}[\Lambda_a U_{\mathcal{C}_{xy}}\Lambda_b U^\dagger_{\mathcal{C}_{xy}}]\mathrm{Tr}(U^\dagger_{\Box_y}\Lambda_b), \tag{5.15}$$

where

$$U_{\mathcal{C}_{xy}} = \mathcal{P}\prod_{(z,\mu)\in\mathcal{C}_{xy}} U_{z,\mu} \tag{5.16}$$

is a path-ordered product of parallel transporters along the open path $\mathcal{C}_{xy}$. This staple-shaped path of time-extent T connects the source and sink points x and y that are spatially separated by a distance R. At these points dynamical $G(2)$ "gluons" are created by plaquette operators $U_{\Box_x}$ and $U_{\Box_y}$. The factors Λ_a and Λ_b reflect the fact that "gluons" transform in the adjoint representation. The closed path symbolic object in the denominator is a Wilson loop of size $R \times 2T$ in the adjoint representation.

The Fredenhagen-Marcu order parameter describes the creation of a pair of adjoint dynamical charges that propagate for a time T and measures their overlap with the vacuum in the limit $R, T \to \infty$. In a Coulomb phase charged states exist in the physical spectrum and are orthogonal to the vacuum. Consequently, the Fredenhagen-Marcu vacuum overlap order parameter then vanishes. In a confined phase, on the other hand, the dynamical charges are screened and $\rho(R,T)$ goes to a non-zero constant for large R and T.

In the strong coupling limit the leading contribution to the numerator of the vacuum overlap order parameter is a tube of plaquettes emanating from the source plaquette $\Box_x$, following the staple-shaped path, and ending at the sink plaquette $\Box_y$. This leads to a perimeter law in the numerator. Just like the Wilson loop in the fundamental representation that was calculated before, the adjoint Wilson loop in the denominator of the order parameter also obeys a perimeter law. Due to the square root and the doubled temporal extent, the perimeter behavior in the numerator and the denominator cancel exactly and one is left with

$$\rho(R,T) = \frac{112(1/7g^2)^{4(2T+R)}}{2(1/7g^2)^{4(2T+R-2)}} = 56(\frac{1}{7g^2})^8 \tag{5.17}$$

Here the plane of the plaquettes $\Box_x$ and $\Box_y$ is dual to the plane of the staple-shaped path $\mathcal{C}_{xy}$. Eq.(5.17) shows that we are indeed in a confined phase (without a string tension, however, with color charge screening) and not in a non-Abelian Coulomb phase. Of course, this strong coupling result does not guarantee that $G(2)$ Yang-Mills theory confines also in the continuum limit, as one would naturally expect. It would be interesting to investigate this issue in numerical simulations.

One might argue that in a pure gluon theory quarks are simply not present and can hence not even be used as external static sources. If one wants to study confinement of gluons without using static quarks, one can use the vacuum overlap operator also in an $SU(3)$ Yang-Mills theory. In the strong coupling limit one then finds again that gluons are in a confined phase with color screening by dynamical gluon creation — and are not in a Coulomb phase.

It should be noted that the Fredenhagen-Marcu order parameter makes sense only at zero temperature, because it requires to take the limit $T \to \infty$. Since $G(2)$ has a trivial center (and thus a vanishing string tension) there is no need for the standard finite temperature deconfinement phase transition. In particular, there is no center symmetry that could break spontaneously at high temperatures. Of course, this argument does not exclude the existence of a first order phase transition at finite temperature. We find it more natural to expect just a crossover. Again, this is an interesting point for numerical investigation. In any case, analytic strong coupling calculations cannot answer this question.

6 Conclusions

We have compared qualitative non-perturbative features such as confinement and chiral symmetry breaking in theories with $G(2)$ and $SU(3)$ gauge groups. In particular, we have exploited the Higgs mechanism (induced by a scalar field in the $\{7\}$ representation of $G(2)$) in order to interpolate smoothly between these two cases. We have focused on effects which are intimately related to the center of the gauge group, and which hence are qualitatively different for $SU(3)$ with center $\mathbf{Z}(3)$ and $G(2)$ with a trivial center.

When all dynamical fields in an $SU(3)$ gauge theory are center-blind (such as gluons or gluinos which have trivial triality) the $\mathbf{Z}(3)$ center is an exact symmetry. Infinitely heavy quarks with non-trivial triality can be used as external probes of the gluon dynamics that provide information about how the $\mathbf{Z}(3)$ symmetry is realized. In a confined phase with intact center symmetry the confining string connecting a static quark-anti-quark pair is absolutely unbreakable and has a non-zero string tension that characterizes the interaction at arbitrarily large distances. The string tension can vanish only when the center gets spontaneously broken, which is indeed unavoidable at high temperatures. Then the Euclidean time extent is short and the gauge field configuration becomes almost static. As a consequence, the Polyakov loop order parameter becomes non-zero. Hence, the exact center symmetry provides us with an argument for the existence of a deconfinement phase transition. If the transition is second order, universality arguments suggest that it is in the universality class of a 3-d center-symmetric scalar field theory for the Polyakov loop [3]. For example, for $N_c = 2$ it is second order [22, 23, 24, 25, 26] and falls in the universality class of the 3-d Ising model [27, 28].

Since it has a trivial center, the concept of triality does not extend to $G(2)$. Consequently, any infinitely heavy external source can be screened by dynamical "gluons" and thus the string always breaks at large distances through the creation of dynamical "gluons". As a result, the string tension ultimately vanishes. However, a strong coupling lattice study of the Fredenhagen-Marcu vacuum overlap order parameter shows that the theory is still confining — i.e. no colored states exist in the spectrum. Confinement without a (fundamental) string tension is indeed exceptional for a pure gauge theory. It only arises for the exceptional Lie groups $G(2)$, $F(4)$ and $E(8)$. As another consequence of the trivial center, the $G(2)$ Polyakov loop is no longer an order parameter. Hence, in contrast to $SU(N_c)$ Yang-Mills theory, for $G(2)$ there is no compelling reason for a finite temperature deconfinement phase transition. We cannot exclude a first order phase transition but we expect only a crossover. Clearly, the triviality of the center implies less predictive power about a possible phase transition.

Once dynamical fields with non-trivial triality (such as light quarks) are included in an $SU(3)$ gauge theory, they break the center symmetry explicitly. As a result, the string connecting a static quark-anti-quark pair can now break through pair creation of dynamical quarks and the string tension ultimately vanishes. Again, the Fredenhagen-Marcu order parameter still signals confinement. In addition, the Polyakov loop is no longer a good order parameter. From this point of view, the confinement in $G(2)$ Yang-Mills theory resembles the one of $SU(3)$ QCD, and hence, it is not so exceptional after all.

Again, by using the Higgs mechanism, we have also interpolated between $G(2)$ and $SU(3)$ gauge theories with massless dynamical fermions, both in the fundamental and in the adjoint representation. In many of these cases, there is a non-trivial chiral symmetry that breaks spontaneously at low temperatures. Since the $G(2)$ representations are real, we have considered N_f flavors of Majorana fermions. The chiral symmetry then is $SU(N_f)_{L=R^*} \otimes \mathbb{Z}(2)_B$ which breaks spontaneously to $SO(N_f)_{L=R} \otimes \mathbb{Z}(2)_B$. It is interesting how this pattern of symmetry breaking turns into the breaking of $SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B$ to $SU(N_f)_{L=R} \otimes U(1)_B$ that occurs in QCD.

As we have seen, there are many interesting non-perturbative phenomena that arise in $G(2)$ gauge theories. Despite the fact that Nature chose not to use $G(2)$ (at least at presently accessible energies) it may be of theoretical interest to study $G(2)$ gauge theories more quantitatively. Lattice gauge theory provides us with a powerful tool for such investigations. For example, it would be interesting to check if the strong coupling confined phase extends to the continuum limit, by measuring the Fredenhagen-Marcu order parameter in a numerical simulation. With lattice methods one can also decide if the low- and high-temperature regimes in $G(2)$ Yang-Mills theory are separated by a first order phase transition or just by a crossover.

It is also interesting to investigate Yang-Mills theories with other gauge groups such as $Sp(N)$, which have a $\mathbb{Z}(2)$ center symmetry. If they have a second order deconfinement phase transition, one expects it to be in the universality class of the 3-d Ising model. A numerical study of $Sp(2)$ gauge theory is presently in progress [53]. The group $Sp(2)$ with 10 generators is the fourth of the rank 2 Lie groups besides $SO(4) \simeq SU(2) \otimes SU(2)$, $SU(3)$ and $G(2)$. Based on its rank and its number of generators one might expect that it should behave more like $SU(3)$ than like $SU(2) = Sp(1)$. However, as in the $SU(2)$ case, we find that $Sp(2)$ Yang-Mills theory has a second order deconfinement phase transition with 3-d Ising critical exponents.

In conclusion, we have used $G(2)$ gauge theories as a theoretical laboratory to study $SU(3)$ theories in an unusual environment. In particular, the embedding of $SU(3)$ in $G(2)$ with its trivial center forces us to think about confinement without the luxury of the $\mathbf{Z}(3)$ symmetry. As one would expect, confinement itself works perfectly well without the center symmetry. However, in its absence we loose predictive power about a possible phase transition at finite temperature.

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A Transition Functions, Twist-Tensor and Consistency Conditions

In this appendix we derive some relations for periodic and C-periodic boundary conditions that are used in section 2. First, we consider periodic boundary conditions. Shifting the gauge field in two orthogonal directions e_ν and e_ρ, on the one hand, one obtains

$$\begin{aligned} A_\mu(x+L_\nu e_\nu+L_\rho e_\rho) &= \Omega_\nu(x+L_\rho e_\rho)(A_\mu(x+L_\rho e_\rho)+\partial_\mu)\Omega_\nu(x+L_\rho e_\rho)^\dagger \\ &= \Omega_\nu(x+L_\rho e_\rho)\Omega_\rho(x)(A_\mu(x)+\partial_\mu)\Omega_\rho(x)^\dagger\Omega_\nu(x+L_\rho e_\rho)^\dagger. \end{aligned} \tag{A.1}$$

On the other hand, by performing the two shifts in the opposite order, one finds

$$\begin{aligned} A_\mu(x+L_\rho e_\rho+L_\nu e_\nu) &= \Omega_\rho(x+L_\nu e_\nu)(A_\mu(x+L_\nu e_\nu)+\partial_\mu)\Omega_\rho(x+L_\nu e_\nu)^\dagger \\ &= \Omega_\rho(x+L_\nu e_\nu)\Omega_\nu(x)(A_\mu(x)+\partial_\mu)\Omega_\nu(x)^\dagger\Omega_\rho(x+L_\nu e_\nu)^\dagger. \end{aligned} \tag{A.2}$$

The two results are consistent only if the transition functions obey the cocycle condition eq.(2.3).

Eq.(2.4) guarantees gauge-covariance of the boundary condition, i.e.

$$\begin{aligned} A_\mu(x+L_\nu e_\nu)' &= \Omega(x+L_\nu e_\nu)(A_\mu(x+L_\nu e_\nu)+\partial_\mu)\Omega(x+L_\nu e_\nu)^\dagger \\ &= \Omega(x+L_\nu e_\nu)\Omega_\nu(x)(A_\mu(x)+\partial_\mu)\Omega_\nu(x)^\dagger\Omega(x+L_\nu e_\nu)^\dagger \\ &= \Omega_\nu(x)'(A_\mu(x)'+\partial_\mu)\Omega_\nu(x)'^\dagger. \end{aligned} \tag{A.3}$$

Interestingly, the gauge transformed cocycle condition takes the form

$$\begin{aligned}
&\Omega_\nu(x+L_\rho e_\rho)'\Omega_\rho(x)' = \Omega_\rho(x+L_\nu e_\nu)'\Omega_\nu(x)'z'_{\nu\rho} \ \Rightarrow \\
&\Omega(x+L_\rho e_\rho+L_\nu e_\nu)\Omega_\nu(x+L_\rho e_\rho)\Omega(x+L_\rho e_\rho)^\dagger\Omega(x+L_\rho e_\rho)\Omega_\rho(x)\Omega(x)^\dagger = \\
&\Omega(x+L_\nu e_\nu+L_\rho e_\rho)\Omega_\rho(x+L_\nu e_\nu)\Omega(x+L_\nu e_\nu)^\dagger\Omega(x+L_\nu e_\nu)\Omega_\nu(x)\Omega(x)^\dagger z'_{\nu\rho} \ \Rightarrow \\
&\Omega_\nu(x+L_\rho e_\rho)\Omega_\rho(x) = \Omega_\rho(x+L_\nu e_\nu)\Omega_\nu(x)z'_{\nu\rho}.
\end{aligned} \tag{A.4}$$

Consequently, $z'_{\nu\rho} = z_{\nu\rho}$, i.e. the twist-tensor is gauge invariant.

Next, we consider C-periodic boundary conditions. As before, we shift the gauge field in two orthogonal directions. First, we pick two different spatial directions i and j, and we obtain

$$\begin{aligned}
A_\mu(x+L_ie_i+L_je_j) &= \Omega_i(x+L_je_j)(A_\mu(x+L_je_j)^*+\partial_\mu)\Omega_i(x+L_je_j)^\dagger \\
&= \Omega_i(x+L_je_j)\Omega_j(x)^*(A_\mu(x)+\partial_\mu)\Omega_j(x)^T\Omega_i(x+L_je_j)^\dagger.
\end{aligned} \tag{A.5}$$

Performing the two shifts in the opposite order, one now finds

$$\begin{aligned}
A_\mu(x+L_je_j+L_ie_i) &= \Omega_j(x+L_ie_i)(A_\mu(x+L_ie_i)^*+\partial_\mu)\Omega_j(x+L_ie_i)^\dagger \\
&= \Omega_j(x+L_ie_i)\Omega_i(x)^*(A_\mu(x)+\partial_\mu)\Omega_i(x)^T\Omega_j(x+L_ie_i)^\dagger.
\end{aligned} \tag{A.6}$$

The two results are consistent only if the transition functions obey the first cocycle condition of eq.(2.14). Next, we pick the spatial i-direction and the Euclidean time direction, such that

$$\begin{aligned}
A_\mu(x+L_ie_i+\beta e_4) &= \Omega_i(x+\beta e_4)(A_\mu(x+\beta e_4)^*+\partial_\mu)\Omega_i(x+\beta e_4)^\dagger \\
&= \Omega_i(x+\beta e_4)\Omega_4(x)^*(A_\mu(x)^*+\partial_\mu)\Omega_4(x)^T\Omega_i(x+\beta e_4)^\dagger.
\end{aligned} \tag{A.7}$$

Again, performing the two shifts in the opposite order we obtain

$$\begin{aligned}
A_\mu(x+\beta e_4+L_ie_i) &= \Omega_4(x+L_ie_i)(A_\mu(x+L_ie_i)+\partial_\mu)\Omega_4(x+L_ie_i)^\dagger \\
&= \Omega_4(x+L_ie_i)\Omega_i(x)(A_\mu(x)^*+\partial_\mu)\Omega_i(x)^\dagger\Omega_4(x+L_ie_i)^\dagger.
\end{aligned} \tag{A.8}$$

In this case, the resulting cocycle condition is the second one of eq.(2.14).

Eq.(2.15) ensures the gauge-covariance of C-periodic boundary condition, i.e.

$$\begin{aligned}
A_\mu(x+L_ie_i)' &= \Omega(x+L_ie_i)(A_\mu(x+L_ie_i)+\partial_\mu)\Omega(x+L_ie_i)^\dagger \\
&= \Omega(x+L_ie_i)\Omega_i(x)(A_i(x)^*+\partial_\mu)\Omega_i(x)^\dagger\Omega(x+L_ie_i)^\dagger \\
&= \Omega_i(x)'(A_\mu(x)'^*+\partial_\mu)\Omega_i(x)'^\dagger.
\end{aligned} \tag{A.9}$$

Let us consider the gauge transformed cocycle condition

$$\begin{aligned}
&\Omega_i(x+L_je_j)'\Omega_j(x)'^* = \Omega_j(x+L_ie_i)'\Omega_i(x)'^* z'_{ij} \;\Rightarrow \\
&\Omega(x+L_je_j+L_ie_i)\Omega_i(x+L_je_j)\Omega(x+L_je_j)^T\Omega(x+L_je_j)^*\Omega_j(x)^*\Omega(x)^\dagger = \\
&\Omega(x+L_ie_i+L_je_j)\Omega_j(x+L_ie_i)\Omega(x+L_ie_i)^T\Omega(x+L_ie_i)^*\Omega_i(x)^*\Omega(x)^\dagger z'_{ij} \;\Rightarrow \\
&\Omega_i(x+L_je_j)\Omega_j(x)^* = \Omega_j(x+L_ie_i)\Omega_i(x)^* z'_{ij}. \qquad \text{(A.10)}
\end{aligned}$$

Hence, $z'_{ij} = z_{ij}$, i.e. the twist-tensor is invariant under the transformations of eq.(2.15). Similarly, we obtain

$$\begin{aligned}
&\Omega_i(x+\beta e_4)'\Omega_4(x)'^* = \Omega_4(x+L_ie_i)'\Omega_i(x)' z'_{i4} \;\Rightarrow \\
&\Omega(x+\beta e_4+L_ie_i)\Omega_i(x+\beta e_4)\Omega(x+\beta e_4)^T\Omega(x+\beta e_4)^*\Omega_4(x)^*\Omega(x)^T = \\
&\Omega(x+L_ie_i+\beta e_4)\Omega_4(x+L_ie_i)\Omega(x+L_ie_i)^\dagger\Omega(x+L_ie_i)\Omega_i(x)\Omega(x)^T z'_{ij} \;\Rightarrow \\
&\Omega_i(x+\beta e_4)\Omega_4(x)^* = \Omega_4(x+L_ie_i)\Omega_i(x) z'_{i4}, \qquad \text{(A.11)}
\end{aligned}$$

such that $z'_{i4} = z_{i4}$.

Interestingly, with C-periodic boundary conditions there are further consistency conditions besides the cocycle condition eq.(2.14). For example, on the one hand, one has

$$\begin{aligned}
&\Omega_i(x+L_je_j+L_ke_k)\Omega_j(x+L_ke_k)^*\Omega_k(x) = \\
&\Omega_j(x+L_ie_i+L_ke_k)\Omega_i(x+L_ke_k)^*\Omega_k(x)z_{ij} = \\
&\Omega_j(x+L_ie_i+L_ke_k)\Omega_k(x+L_ie_i)^*\Omega_i(x)z_{ij}z_{ki} = \\
&\Omega_k(x+L_ie_i+L_je_j)\Omega_j(x+L_ie_i)^*\Omega_i(x)z_{ij}z_{ki}z_{jk}, \qquad \text{(A.12)}
\end{aligned}$$

while, on the other hand,

$$\begin{aligned}
&\Omega_i(x+L_je_j+L_ke_k)\Omega_j(x+L_ke_k)^*\Omega_k(x) = \\
&\Omega_i(x+L_je_j+L_ke_k)\Omega_k(x+L_je_j)^*\Omega_j(x)z_{kj} = \\
&\Omega_k(x+L_je_j+L_ie_i)\Omega_i(x+L_je_j)^*\Omega_j(x)z_{kj}z_{ik} = \\
&\Omega_k(x+L_je_j+L_ie_i)\Omega_j(x+L_ie_i)^*\Omega_i(x)z_{kj}z_{ik}z_{ji}. \qquad \text{(A.13)}
\end{aligned}$$

Hence, unlike for periodic boundary conditions, there is the constraint of eq.(2.16), $z_{ij}^2 z_{jk}^2 z_{ki}^2 = 1$, on the twist-tensor itself. Similarly, if one shifts in two spatial directions as well as in the Euclidean time direction, on the one hand, one finds

$$\begin{aligned}
&\Omega_i(x+L_je_j+\beta e_4)\Omega_j(x+\beta e_4)^*\Omega_4(x) = \\
&\Omega_j(x+L_ie_i+\beta e_4)\Omega_i(x+\beta e_4)^*\Omega_4(x)z_{ij} = \\
&\Omega_j(x+L_ie_i+\beta e_4)\Omega_4(x+L_ie_i)^*\Omega_i(x)^*z_{ij}z_{i4}^* = \\
&\Omega_4(x+L_ie_i+L_je_j)\Omega_j(x+L_ie_i)\Omega_i(x)^*z_{ij}z_{i4}^*z_{j4}, \qquad \text{(A.14)}
\end{aligned}$$

while, on the other hand,

$$\Omega_i(x+L_je_j+\beta e_4)\Omega_j(x+\beta e_4)^*\Omega_4(x) =$$

$$\begin{aligned}
&\Omega_i(x+L_je_j+\beta e_4)\Omega_4(x+L_je_j)^*\Omega_j(x)^*z_{j4}^* = \\
&\Omega_4(x+L_je_j+L_ie_i)\Omega_i(x+L_je_j)\Omega_j(x)^*z_{j4}^*z_{i4} = \\
&\Omega_4(x+L_je_j+L_ie_i)\Omega_j(x+L_ie_i)\Omega_i(x)^*z_{j4}^*z_{i4}z_{ij}.
\end{aligned} \tag{A.15}$$

Consequently, one also obtains eq.(2.17), $z_{i4}^2 = z_{j4}^2$.

References

[1] A. M. Polyakov, Phys. Lett. B72 (1978) 477.

[2] L. Susskind, Phys. Rev. D20 (1979) 2610.

[3] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B210 (1982) 423; Phys. Rev. D26 (1982) 963.

[4] K. Holland and U.-J. Wiese, in "At the frontier of particle physics — handbook of QCD", edited by M. Shifman, World Scientific, 2001 [arXiv:hep-ph/0011193].

[5] J. Greensite, arXiv:hep-lat/0301023.

[6] J. Greensite and B. Lautrup, Phys. Rev. Lett. 47 (1981) 9.

[7] G. Bhanot and M. Creutz, Phys. Rev. D24 (1981) 3212.

[8] I. G. Halliday and A. Schwimmer, Phys. Lett. B101 (1981) 327; Phys. Lett. B102 (1981) 337.

[9] L. Caneschi, I. G. Halliday and A. Schwimmer, Nucl. Phys. B200 (1982) 409.

[10] S. Cheluvaraja and H. S. Sharatchandra, arXiv:hep-lat/9611001.

[11] S. Datta and R. V. Gavai, Phys. Rev. D57 (1998) 6618 [arXiv:hep-lat/9708026].

[12] P. de Forcrand and O. Jahn, Nucl. Phys. B651 (2003) 125 [arXiv:hep-lat/0211004].

[13] H. Reinhardt and T. Tok, Phys. Lett. B500 (2001) 173 [arXiv:hep-th/0011068].

[14] E. Hilf and L. Polley, Phys. Lett. B131 (1983) 412.

[15] G. 't Hooft, Nucl. Phys. B138 (1978) 1; Nucl. Phys. B153 (1979) 141.

[16] L. Polley and U.-J. Wiese, Nucl. Phys. B356 (1991) 629.

[17] A. S. Kronfeld and U.-J. Wiese, Nucl. Phys. B357 (1991) 521; Nucl. Phys. B401 (1993) 190 [arXiv:hep-lat/9210008].

[18] U.-J. Wiese, Nucl. Phys. B375 (1992) 45.

[19] M. Gunaydin and F. Gursey, J. Math. Phys. 14 (1973) 1651.

[20] W. G. McKay, J. Patera and D. W. Rand, "Tables of representations of simple Lie algebras", Vol. I: Exceptional simple Lie algebras, Les Publications CRM, Montréal.

[21] K. Fredenhagen and M. Marcu, Phys. Rev. Lett. 56 (1986) 223.

[22] L. D. McLerran and B. Svetitsky, Phys. Rev. D24 (1981) 450; Phys. Lett. B98 (1981) 195.

[23] J. Kuti, J. Polonyi and K. Szlachanyi, Phys. Lett. B98 (1981) 199.

[24] J. Engels, F. Karsch, H. Satz and I. Montvay, Phys. Lett. B101 (1981) 89.

[25] R. V. Gavai, Nucl. Phys. B215 (1983) 458.

[26] R. V. Gavai, F. Karsch and H. Satz, Nucl. Phys. B220 (1983) 223.

[27] J. Engels, J. Fingberg and M. Weber, Nucl. Phys. B332 (1990) 737.

[28] J. Engels, J. Fingberg and D. E. Miller, Nucl. Phys. B387 (1992) 501.

[29] T. Celik, J. Engels and H. Satz, Phys. Lett. B125 (1983) 411.

[30] J. B. Kogut, M. Stone, H. W. Wyld, W. R. Gibbs, J. Shigemitsu, S. H. Shenker and D. K. Sinclair, Phys. Rev. Lett. 50 (1983) 393.

[31] S. A. Gottlieb, J. Kuti, D. Toussaint, A. D. Kennedy, S. Meyer, B. J. Pendleton and R. L. Sugar, Phys. Rev. Lett. 55 (1985) 1958.

[32] F. R. Brown, N. H. Christ, Y. F. Deng, M. S. Gao and T. J. Woch, Phys. Rev. Lett. 61 (1988) 2058.

[33] R. V. Gavai, F. Karsch and B. Petersson, Nucl. Phys. B322 (1989) 738.

[34] M. Fukugita, M. Okawa and A. Ukawa, Phys. Rev. Lett. 63 (1989) 1768.

[35] N. A. Alves, B. A. Berg and S. Sanielevici, Phys. Rev. Lett. 64 (1990) 3107.

[36] S. J. Knak Jensen and O. G. Mouritsen, Phys. Rev. Lett. 43 (1979) 1736.

[37] H. W. J. Blöte and R. H. Swendsen, Phys. Rev. Lett. 43 (1979) 799.

[38] H. J. Herrmann, Z. Phys. B35 (1979) 171.

[39] F. Y. Wu, Rev. Mod. Phys. 54 (1982) 235.

[40] M. Fukugita and M. Okawa, Phys. Rev. Lett. 63 (1989) 13.

[41] P. Hasenfratz, F. Karsch and I. O. Stamatescu, Phys. Lett. B133 (1983) 221.

[42] F. Karsch and S. Stickan, Phys. Lett. B488 (2000) 319 [arXiv:hep-lat/0007019].

[43] R. D. Pisarski and F. Wilczek, Phys. Rev. D29 (1984) 338.

[44] E. Witten, Nucl. Phys. B507 (1997) 658 [arXiv:hep-th/9706109].

[45] A. Campos, K. Holland and U.-J. Wiese, Phys. Rev. Lett. 81 (1998) 2420 [arXiv:hep-th/9805086]; Phys. Lett. B443 (1998) 338 [arXiv:hep-th/9807207].

[46] Z. Frei and A. Patkos, Phys. Lett. B229 (1989) 102.

[47] T. Trappenberg and U.-J. Wiese, Nucl. Phys. B372 (1992) 703.

[48] M. Gross and J. F. Wheater, Nucl. Phys. B240 (1984) 253.

[49] A. Gocksch and M. Okawa, Phys. Rev. Lett. 52 (1984) 1751.

[50] G. G. Batrouni and B. Svetitsky, Phys. Rev. Lett. 52 (1984) 2205.

[51] M. Wingate and S. Ohta, Phys. Rev. D63 (2001) 094502 [arXiv:hep-lat/0006016].

[52] B. Lucini, M. Teper and U. Wenger, Phys. Lett. B545 (2002) 197 [arXiv:hep-lat/0206029].

[53] K. Holland, M. Pepe and U.-J. Wiese, in preparation.